III. Least Squares

A. Network
B. Adjusting a Network
C. Redundancies
D. Least Squares Adjustment
E. Pin Cushions

III. Least Squares

A. Network

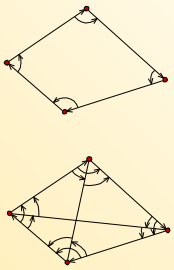
A network consists one or more directly and/or indirectly measured quantities.

Simple network: Sometimes we know the *total* error:

$$\sum(\text{int. angles}) = (n-2)\times 180^\circ$$

$$\sum \text{Lat} = 0 \qquad \sum \text{Dep} = 0$$

Complex network: Or there are so many measurements of varying quality that determining a "total" error is difficult or impossible.



III. Least Squares

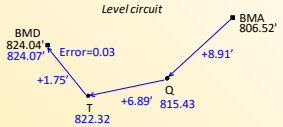
B. Adjusting a Network

Adjustment: process of distributing error.

Simple network usually adjusted using a basic mathematical model.

Example: Each point has a single "raw" elevation.
Use an even distribution

- Closure error at BM D = 814.07-824.04 = 0.03 high
- Corr'n per elev = -0.03/3 = -0.01
- Adj Elev_Q = 815.43+(1)(-0.01) = 815.42
- Adj Elev_T = 822.32+(2)(-0.01) = 822.30
- Adj $\text{Elev}_{BM\,D}$ = 824.07+(3)(-0.01) = 824.04 *check*
- Each point has a single adjusted elevation

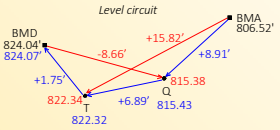


III. Least Squares

B. Adjusting a Network

Adjustment: process of distributing error.

- Add runs between non-adjacent points.
- Points Q and T have multiple raw elevations.
- How to apply simple adjustment?
- Will Q & T end up with a single adjusted elevation each?



Have to use a "best-fit" adjustment model.

- An LSA minimizes the sum of the squares of the residuals of the observations: $\Sigma(v^2)$=min

III. Least Squares

B. Adjusting a Network

Simple Adjustment

- Advantages
  - Easy
- Disadvantages
  - Treats random errors systematically
  - Can't determine quality of adjusted value

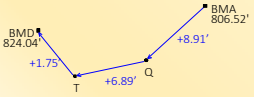
Least Squares

- Advantages
  - Models random errors better
  - Able to deal with multiple unknowns simultaneously
  - Easily incorporate redundant measurements
  - Mix different quality measurements
  - Can generate statistics for
    - overall adjustment
    - individual unknowns
- Disadvantages
  - Computation intensive
  - Statistics overload
  - Easy to misuse

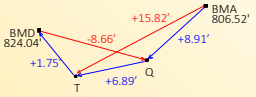
III. Least Squares

C. Redundancies

1. Vertical
   - Redundancy; aka Degree of freedom (df)
     - df = m-n
       - m: number of measurements
       - n: number of unknowns



m 3 BMA-Q, Q-T, T-BMD
n 2 Q, T
df 1



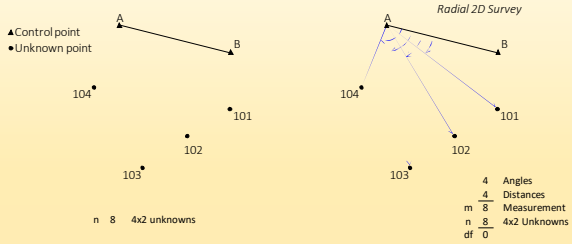
m 5 BMA-Q, Q-T, T-BMD, BMD-Q, BMA-T
n 2 Q, T
df 3

III. Least Squares

C. Redundancies

2. Horizontal

Each point in a horizontal network has two unknowns: N and E.



Radial 2D Survey

n 8 4x2 unknowns

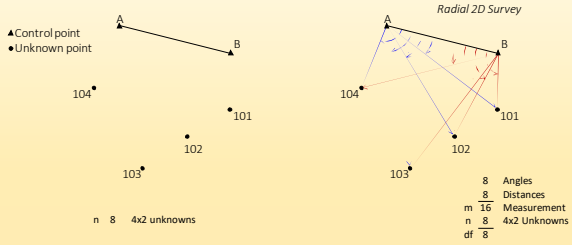
	4	Angles
	4	Distances
m	8	Measurement
n	8	4x2 Unknowns
df	0	

III. Least Squares

C. Redundancies

2. Horizontal

Each point in a horizontal network has two unknowns: N and E.



Radial 2D Survey

n 8 4x2 unknowns

	8	Angles
	8	Distances
m	16	Measurement
n	8	4x2 Unknowns
df	8	

III. Least Squares

D. Least Squares Adjustment

1. Fundamentals

Random errors only

Mistakes found and removed

Systematic errors compensated

Correct weighting scheme

A priori estimates

Equipment MSA

Experience

An LSA best-fits measurements to control points by distributing random errors into them.

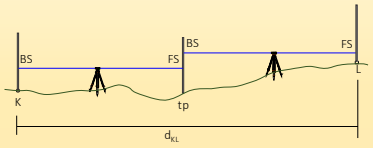
Difference between an adjusted and original measurement is its residual.

LSA minimizes thee sum of the squares of the residuals: $\Sigma(v^2)=min$

III. Least Squares

D. Least Squares Adjustment

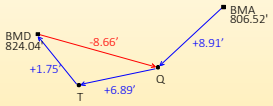
2. Vertical
   - In leveling
     - Unknowns are elevations
     - Measurements are simple addition and subtraction
       - $Elev_B = Elev_A + BS_A - FS_B$
     - Simple math means LSA is a direct solution of simultaneous equations.



III. Least Squares

D. Least Squares Adjustment

2. Vertical
   - To minimize $\Sigma(v^2)$, an equation including a residual must be created for each measurement.

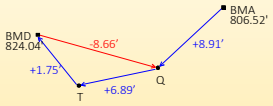


Line		
BMA-Q:	$Elev_Q = 806.52 + 8.91 + v_{aq}$	$v_{aq} = Elev_Q - 815.43$
Q-T:	$Elev_T = Elev_Q + 6.89 + v_{qt}$	$v_{qt} = Elev_T - Elev_Q - 6.89$
T-BMD:	$824.04 = Elev_T + 1.75 + v_{tb}$	$v_{tb} = 822.29 - Elev_T$
BMD-Q	$Elev_Q = 824.04 - 8.66 + v_{bq}$	$v_{bq} = Elev_Q - 815.38$

III. Least Squares

D. Least Squares Adjustment

2. Vertical
   - To minimize $\Sigma(v^2)$, an equation including a residual must be created for each measurement.



Square each residual equation and add them

$$F = \sum_{1}^{n} v_i^2 = (Elev_Q - 815.43)^2 + (Elev_T - Elev_Q - 6.89)^2 + (822.29 - Elev_T)^2 + (Elev_Q - 815.38)^2$$

The function must be minimized for each unknown.
Take the partial derivative with respect to each unknown elevation and set equal to 0.

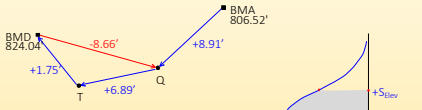
$$\frac{\partial F}{\partial Elev_Q} = 0.000 \qquad \frac{\partial F}{\partial Elev_T} = 0.000$$

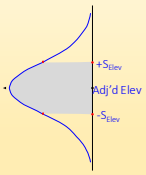
III. Least Squares

D. Least Squares Adjustment

2. Vertical

Statistics: Once elevations are determined, residuals can be computed and then standard errors.





III. Least Squares

D. Least Squares Adjustment

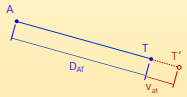
3. Horizontal

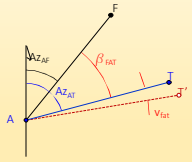
Unknowns are coordinates

Measurements are angles and distances.

Position determination requires trig which is non-linear.

Partial derivatives with trig functions is a pain.





$$v_{at} = D_{AT} - \left[(N_T - N_A)^2 + (E_T - E_A)^2 \right]^{1/2}$$

$$v_{fat} = \beta_{fAT} - (Az_{AT} - Az_{AF}) = \beta_{fAT} - \left(\tan^{-1}\left[\frac{E_T - E_A}{N_T - N_A}\right] - \tan^{-1}\left[\frac{E_F - E_A}{N_F - N_A}\right] \right)$$

III. Least Squares

D. Least Squares Adjustment

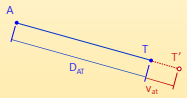
3. Horizontal

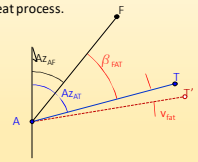
Solution is iterative:

Start with initial coordinate approximations

Use minimization function to compute coordinate corrections

If corrections are significant, apply to coordinates & repeat process.





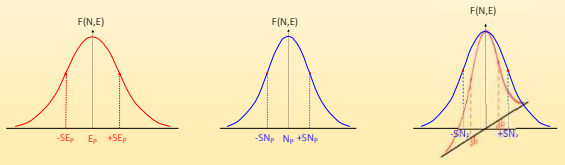
$$v_{at} = D_{AT} - \left[(N_T - N_A)^2 + (E_T - E_A)^2 \right]^{1/2}$$

$$v_{fat} = \beta_{fAT} - (Az_{AT} - Az_{AF}) = \beta_{fAT} - \left(\tan^{-1}\left[\frac{E_T - E_A}{N_T - N_A}\right] - \tan^{-1}\left[\frac{E_F - E_A}{N_F - N_A}\right] \right)$$

III. Least Squares

D. Least Squares Adjustment

3. Horizontal
   Statistics
   Each adjusted coordinate of a pair has its own standard error: S_N and S_E

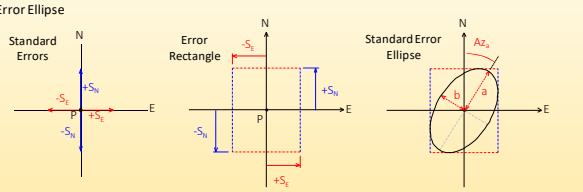


Bi-variate Distribution

III. Least Squares

D. Least Squares Adjustment

3. Horizontal
   Error Ellipse

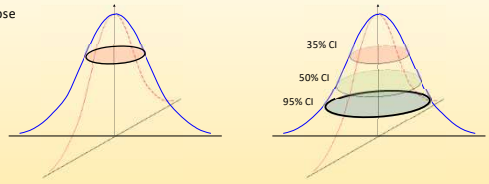


The position is weakest in the direction of the semi-major axis, a. Its direction is Az_a.
Position is strongest in the semi-minor axis, b, direction.

III. Least Squares

D. Least Squares Adjustment

3. Horizontal
   Error Ellipse



Standard error ellipse represents approx. 33-35% of the volume under the bivariate distribution surface.
33-35% CI

95% CI error ellipse uses approx. 2.45 multiplier on the semi-major and –minor axes.
(Caution: Not all software use the same multiplier to get 95% CI)

III. Least Squares

E. Pin Cushion

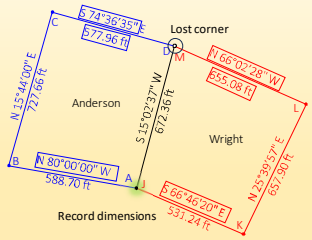
A pincushion is a corner with multiple monuments set in close proximity to each other.

Not limited to land surveyors…

III. Least Squares

E. Pin Cushion

Example: Independent surveys of two lots



Properties share common boundary.
All corners exist except common corner at D/M

2021 Jones surveys Anderson property
- Starts at A, uses record bearing to B, measures to C, and uses record distance and bearing to D.

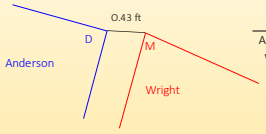
2023 Mills surveys Wright property.
- Starts at J, uses record bearing to K, measures to L, uses record distance and bearing to M

Both surveys adjusted in *StarNet* using a 95% CI

III. Least Squares

E. Pin Cushion

Example: Independent surveys of two lots



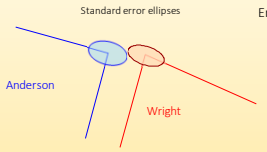
95% CI Adjustment results

Parcel	*North, ft*	*East, ft*	S_N	S_E
Anderson, D	2649.294	1174.077	0.138	0.221
Wright, M	2649.267	1174.507	0.121	0.209

III. Least Squares

E. Pin Cushion

Example: Independent surveys of two lots



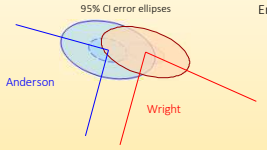
Error ellipses

Point	a	b	Az_a	
D (Anderson)	0.225	0.130	104°19′	Standard
M (Wright)	0.217	0.106	100°00′	Standard

III. Least Squares

E. Pin Cushion

Example: Independent surveys of two lots



Error ellipses

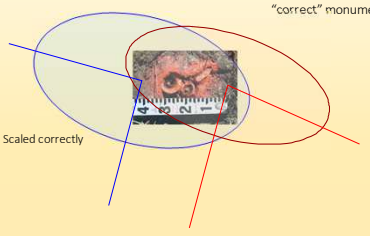
Point	a	b	Az_a	
D (Anderson)	0.225 0.552	0.130 0.319	104°19′	Standard 95% CI
M (Wright)	0.217 0.531	0.106 0.260	100°00′	Standard 95% CI

III. Least Squares

E. Pin Cushion

Example: Independent surveys of two lots

Should each surveyor have placed their own "correct" monument?



III. Least Squares

E. Pin Cushion

Understanding error behavior helps avoid creating problems like this:

Commencing at a 2" iron pipe at the west quarter corner of Section 31, T5N, R10E;
Thence S88°16'52"W, 0.30 feet to the existing east line of Section 1, T5N, R9E;
Thence S00°18'01"W, 0.01 feet along said east line of said Section 1;
Thence S00°18'01"W, 33.20 feet along said east line;
Thence N88°34'15"E, 33.78 feet to the existing east right-of-way line of STH 104, also being the point of beginning;
Thence N88°47'53"E, 803.03 feet along the existing south right-of-way line of STH 92;
Thence N88°17'20"E, 55.46 feet along said south right-of-way line;